Lecture

# Taylor & MacLaurin Series (11.10)

08/03/2017

recall: power series

eg. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ , $f(x) = \frac{1}{1-x}$

for $|x| < 1$

today: systematic way of finding the power series of a function

**Theorem** If $f$ has a power series representation at a point $x=a$:
then, its coefficients are given by its derivatives:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

remark: $f^{(0)}(a) = f(a)$, if $n=0$: $0! = 1$ by default

**Ex** Taylor series of $f(x) = e^{2x}$ at $a=1$

$f'(x) = e^{2x} \cdot 2$

$f''(x) = (e^{2x} \cdot 2) \cdot 2$

$\vdots$

$\boxed{f^{(n)}(x) = 2^n \cdot e^{2x}}$ in general → don't need to prove, just identify

at $a=1$

$f'(1) = 2e^2$

$f''(1) = 2^2 \cdot e^2$

$\boxed{f^{(n)}(1) = 2^n \cdot 2^2}$

PUT TOGETHER

$$f(x) = \sum_{n=0}^{\infty} \frac{\overbrace{2^n \cdot e^2}^{f^{(n)}(1)}}{n!}(x-1)^n$$

**Ex** $f(x) = \sin(x)$ at $a=0$

A Taylor Series for $f(x)$ at $x=a=0$, is called a $\boxed{\text{MacLaurin series}}$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$ $(= f(x))$

We only have 4 different derivatives:

$f^{(4k)}(x) = \sin(x)$

$f^{(4k+1)}(x) = \cos(x)$

$f^{(4k+2)}(x) = -\sin(x)$

$f^{(4k+3)}(x) = -\cos(x)$

EVALUATE AT $a=0$: $f^{(4k)}(0) = \sin(0) = 0$

$f^{(4k+1)}(0) = \cos(0) = 1$

$f^{(4k+2)}(0) = -\sin(0) = 0$

$f^{(4k+3)}(0) = -\cos(0) = -1$

$$f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= \sum_{k=0}^{\infty} \left( \frac{\overbrace{f^{(4k)}(0)}^{0}}{(\frac{k}{4})!} x^{k/4} + \frac{\overbrace{f^{(4k+1)}(0)}^{1}}{(\frac{k}{4}+1)!} x^{k/4+1} + \frac{\overbrace{f^{(4k+2)}(0)}^{0}}{(\frac{k}{4}+2)!} x^{\frac{k}{4}+2} + \frac{\overbrace{f^{(4k+3)}(0)}^{-1}}{(\frac{k}{4}+3)!} x^{\frac{k}{4}+3} \right)$$

($\frac{k}{4}$ plays the role of $n$)

$$= \sum_{k=0}^{\infty} \left( \frac{1}{(\frac{k}{4}+1)!} x^{k/4+1} + \frac{-1}{(\frac{k}{4}+3)!} x^{k/4+3} \right)$$

It's enough to just look at the odd terms: use $2n+1$ instead of $k$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$ ← remember!

# Important series to remember

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$ , geometric series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

☆ NB

see textbook for radius of convergence & more

Ex Maclaurin Series for $f(x) = \sin(x^2)$ (recall $a=0$)

$$f'(x) = (\sin(x^2))' = \cos(x^2) \cdot 2x$$

$$f''(x) = -\sin(x^2) \cdot 2x \cdot 2x + 2\cos(x^2)$$

The terms for $f^{(n)}(x)$ become very long and hard to manage.

Instead: use substitution!

We know $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Substitute $x^2$ for $x$ in the series for $\sin(x)$.

$$\boxed{\sin(x^2)} = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}}$$

$$\sin(x^2) = \frac{x^2}{1} + \frac{x^4}{6} + \dots$$

**Ex** Find a power series for $f(x) = \ln(1+x)$ and its radius of convergence.

Choose a MacLaurin series.

$$f'(x) = \frac{1}{1+x} \qquad f'''(x) = -\frac{-2}{(1+x)^3} = \frac{2}{(1+x)^3}$$

$$f''(x) = \frac{-1}{(1+x)^2} \qquad f^{(4)}(x) = \frac{-2\cdot 3}{(1+x)^4}$$

watch signs

In general: $$f^{(n)}(x) = \frac{(n-1)!\cdot(-1)^{n+1}}{(1+x)^n}$$

Evaluate at $a=0$:

same as $n+1$, to match $(n-1)!$

$$f^{(n)}(0) = \frac{(n-1)!(-1)^{n+1}}{(1+0)^n} = (n-1)!(-1)^{n-1}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{\overbrace{(n-1)!(-1)^{n-1}}^{\text{derivative}}}{\underbrace{n!}_{\text{from formula}}} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot x^n}{n} \qquad \boxed{n! = (n-1)!\cdot n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)(-1)^n \cdot x^n}{n} = \boxed{\sum_{n=0}^{\infty} \frac{-(-x)^n}{n}}$$

**Remember, the MacLaurin series for $f(x)$ is $\sum_{n=}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$**

Now use a trick for the same result

we know $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

integrate: $$\int \frac{1}{1-x}\,dx = \int \left(\sum_{n=0}^{\infty} x^n\right) dx$$

$$-\ln|1-x| = \sum_{n=0}^{\infty} \int x^n\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Use substitution again: want $\ln(1+x)$.

we have $$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

To get the series for $\ln(1+x)$, we substitute $-x$ for $x$.

$$-\ln|1-(-x)| = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}$$

$$\ln(1+x) = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} = \boxed{+\sum_{n=1}^{\infty} \frac{-(-x)^n}{n}}$$

Last step radius of convergence of $\sum_{n=1}^{\infty} \frac{-(-x)^n}{n}$

root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{-(-x)^n}{n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{x^n}{n}}$$

$$= \lim_{n\to\infty} \frac{x}{\underbrace{\sqrt[n]{n}}_{\to 1}} = x$$

root test: if $L < 1$ : series converges

If $|x| < 1$, the series converges.

The radius of convergence is 1.

What happens at $x = -1$ and $x = 1$?

$\underline{x = -1}$ $\quad \sum_{n=1}^{\infty} \frac{-(-(-1))^n}{n} = \sum_{n=1}^{\infty} \frac{-(1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} \boxed{= -\sum_{n=1}^{\infty} \frac{1}{n}}$

This is the harmonic series which diverges.

$\underline{x = 1}$ $\quad \sum_{n=1}^{\infty} \frac{-(-1)^n}{n} \boxed{= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}} \rightarrow$ Alternating harmonic series which converges.

The interval of convergence is $-1 < x \leq 1$, or $(-1, 1]$